

**METHOD OF APPLICATION OF MOMENT
DISTRIBUTION TO THE SOLUTION OF
ARCHED BENTS WITH A
VARYING MOMENT OF INERTIA**

JOHN MASCENIK

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METHOD OF APPLICATION OF MOMENT DISTRIBUTION TO THE
SOLUTION OF ARCHED BENTS WITH A VARYING MOMENT OF INERTIA

By

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INTRODUCTION

TABLE OF CONTENTS

INTRODUCTION	PAGE
Object	1
Definitions of Symbols	2
PROCEDURE	
Introduction	4
Fixed-End Moment and Thrust Determination.	5
Determination of Carry-over Factor	9
Absolute Stiffness	11
Spread	12
RESULTS	
Conclusions.	13
Results Based on a Unit Span	14
APPLICATION OF RESULTS	15
APPENDIX	
Appendix A - Tabulation of Arc Lengths . .	22
Appendix B - Moment of Inertia I_x and Rel. $1/I_x$	24
Appendix C - Tabulation of Elastic Areas and Centroid of Arch.	27
Appendix D - Solution of Arched Bent for M_L , M_R , H , and V	31
Appendix E - Sample Calculations for De- termining Absolute Stiffness, Carry-over Factor, Horizontal Thrust due to Rotation of the Joint, and the Effect of Spread	42
BIBLIOGRAPHY	48

OBJECT

This work is an extension of the thesis of Cain¹ and Hansen and Jones² in the application of moment distribution to the solution of arched bents. This thesis will, however, deal with an arched bent of varying moment of inertia.

The fixed end moments for an arched bent whose moment of inertia varies as the secant of the angle α has been derived by Iselin and La Lande³. It will be the object of this thesis to correlate these moments and other constants as found in this thesis in order to present a solution of arched bents by moment distribution.

¹Cain, M.G., Application of Moment Distribution to Arched Bents,
Thesis, R.P.I., February, 1947

²B. L. Hansen and R. O. Jones, Method of Application of Moment Distribution to Solution of Arched Bents, Thesis, R.P.I., May, 1948

³D. G. Iselin and A. M. La Lande, A Development of an Exact Method for the Solution of Symmetrical Fixed End Parabolic Arches,
Thesis, R.P.I., August, 1948

DEFINITIONS OF SYMBOLS

S or L	-Length of span in feet.
h	-Rise in feet - distance from springing line to crown.
H	-Horizontal thrust in kips.
M _R	-Fixed end moment in kips at right springing.
M _L	-Fixed end moment in kips at left springing.
V	-Vertical reaction.
a	- $\frac{4h}{L^2}$
y	- ax^2 , equation of parabolic arch.
I _x	-Cross sectional moment of inertia at any point along the arch.
I _c	-Cross sectional moment of inertia at the center of the arch.
alpha	-Angle between tangent to working line of arch, and a line parallel to springing line.
ds	-Increment of length along the arch itself.
dx	-Increment of length parallel to X-axis (along the springing line).
DELTA H	-Total horizontal deflection of a point.
DELTA V	-Total vertical deflection of a point.
THETA M	-Total angular rotation of a point.
delta hv	-Horizontal deflection of a point due to the application of a unit vertical force.
delta hm	-The linear horizontal deflection caused by a unit moment.
delta vm	-Vertical deflection of a point due to the application of a unit moment.
delta hh	-The linear deflection of a point caused by a unit horizontal load at the point.

δv -Vertical deflection of a point due to the application of a unit vertical force.

α_{mv} -Angular rotation of the member at a point due to the application of a unit vertical force.

α_{mh} -Angular rotation of the member at a point due to the application of a unit horizontal force.

α_{mm} - Angular rotation of the member at a point due to the application of a unit moment.

$\frac{ds}{rel.I} = A$ -A relative area of the arc segment.

x -The horizontal distance from a reference point, usually the left end of the arch.

x' -The horizontal distance from the neutral point to the point on the arch where the M/EI loading on the arch segment can be considered concentrated.

y -The vertical distance from a reference point which is usually the left springing.

y' -The vertical distance from the neutral point to the point on the arch where the M/EI loading can be considered concentrated.

$\bar{y}$ -The vertical distance from the springing line to the neutral point.

PROCEDURE

PROCEDURE

Introduction

Indeterminate rigid frames can be solved by moment distribution whether the members are of constant or varying cross section, curved or straight, provided the constants (a) stiffness at each end, (b) fixed-end moments at each end, and (c) carry-over factor at each end, are known or can be determined.

It is the purpose of this thesis to provide the constants required to solve a rigid frame whose curved top member's moment of inertia varies as the secant of the angle α for various rise/span ratios.

In this work a parabolic arch whose moment of inertia varies with the secant of the angle α and whose rise/span values are 0.04, 0.08, 0.20, 0.30, and 0.40, are dealt with. Solutions were obtained for 20 and 30 foot spans. All tabulated results are based upon a unit span of 1 foot.

The fixed-end moments as tabulated were found by using the exact solution as determined by Iselin and La Lande in their thesis entitled "A Development of an Exact Method for the Solution of Symmetrical Fixed End Parabolic Arches". In order to see of what magnitude the results which were obtained from using an approximate solution varied from the exact values, a few rise/span ratios were solved using a combination of the Conjugate Structure Method and the Neutral Point Method. The computation of the carry-over factors, absolute stiffness, and the effect of spread were made by using the General Method and Virtual Work.

The results were then combined and a sample problem was worked to illustrate the procedure in using the results obtained.

Fixed-End Moment and Thrust Determination

In the determination of the fixed-end moments, use was made of the exact solution method derived by Iselin and La Lande¹ for the solution of the parabolic arch whose moment of inertia varied as the secant of the angle alpha.

The formulae used were as follows:

$$H = \frac{15L}{8h} \left(\frac{1}{8} - v^2 + 2v^4 \right)$$

$$M_R = \frac{L}{4} \left(\frac{1}{8} + v - 3v^2 - 4v^3 + 10v^4 \right)$$

$$M_L = \frac{L}{4} \left(\frac{1}{8} - v - 3v^2 + 4v^3 + 10v^4 \right)$$

L = Length of span

$v = \frac{W}{L}$, ratio of unit load off-center distance to length of span

w = Distance in feet from centerline to unit load

The above formulae for M_R and M_L are valid only when the unit load is placed to the left of the center or at the center. When the load is to the right of center, interchange M_R and M_L in the above equations.

The results of the above equations can be found in the section entitled "Results".

In order to determine the magnitude of error involved in using the approximate solution, and to show the method of analyzing arched bents which cannot be solved by exact mathematical formulae, the following solution is presented.

¹See Bibliography

In the determination of the fixed-end moments, it was decided to use a combination of the Neutral Point Method and the Conjugate Structure Method, thereby eliminating the need of solving simultaneous equations.

The arch is cut at the left reaction, cantilevering out from the right end, and a weightless bracket of infinite cross section rigidly connects the cut end with the neutral point.

The arch is divided into ten equal horizontal projections. The arc length of each as solved by Hansen and Jones¹ are presented in Appendix A. It was then necessary to assume that the moment of inertia at the midpoint of the arc segments was the average moment of inertia for that segment. Computations of the moments of inertia are presented in Appendix B.

Since the elastic areas vary with the arc lengths and the average moment of inertia of the arc segments, it was necessary to compute these elastic areas. This was done by using the moment of inertia at the center of the arch, I_c , as the reference moment of inertia and computing the relative I_c/I_x for the segment. This corresponds to the width of the conjugate structure. The product of the relative I_c/I_x and the arc segment length gives the elastic area of that segment.

The Conjugate Structure Method mentioned above was developed by J. Sterling Kinney, Professor of Structural Engineering, R.P.I., Troy, New York. Its method of application is as described below.

"The conjugate structure, for a given real structure, is identical to the real structure with regards to the lengths of the members and their relative positions. The conjugate

¹See Bibliography

structure is positioned in a horizontal plane; and the load, which acts in a vertical direction on the conjugate structure, is the M/EI diagram of the real structure. That is, the conjugate structure is loaded with the flexural strains of the real structure.

"The end of the conjugate structure, corresponding to the end of the real structure which suffers deflection, always has a fixed support.

"The internal moment on any section of the conjugate structure is the deflection of the corresponding section of the real structure in a direction perpendicular to the lever arm used to find any particular moment.

"The shear at any section of the conjugate structure is the slope of the corresponding section of the real structure.

"Sign Convention for the Conjugate Structure

"If the flexural strain at a given section of the real structure is such as to cause tension on the outside fibers, then this flexural strain is represented as a downward load on the conjugate structure. If compression exists on the outside fibers of the real structure, the load on the corresponding section of the conjugate structure is up.

"If the moment of any section of the conjugate structure results in tension in the top fiber, the vertical deflection at the corresponding section of the real structure is down and the horizontal deflection is such as to shorten the horizontal projection of the distance between given points in the structure.

"If a section be passed through any point of the conjugate structure and if the portion of the conjugate structure to the

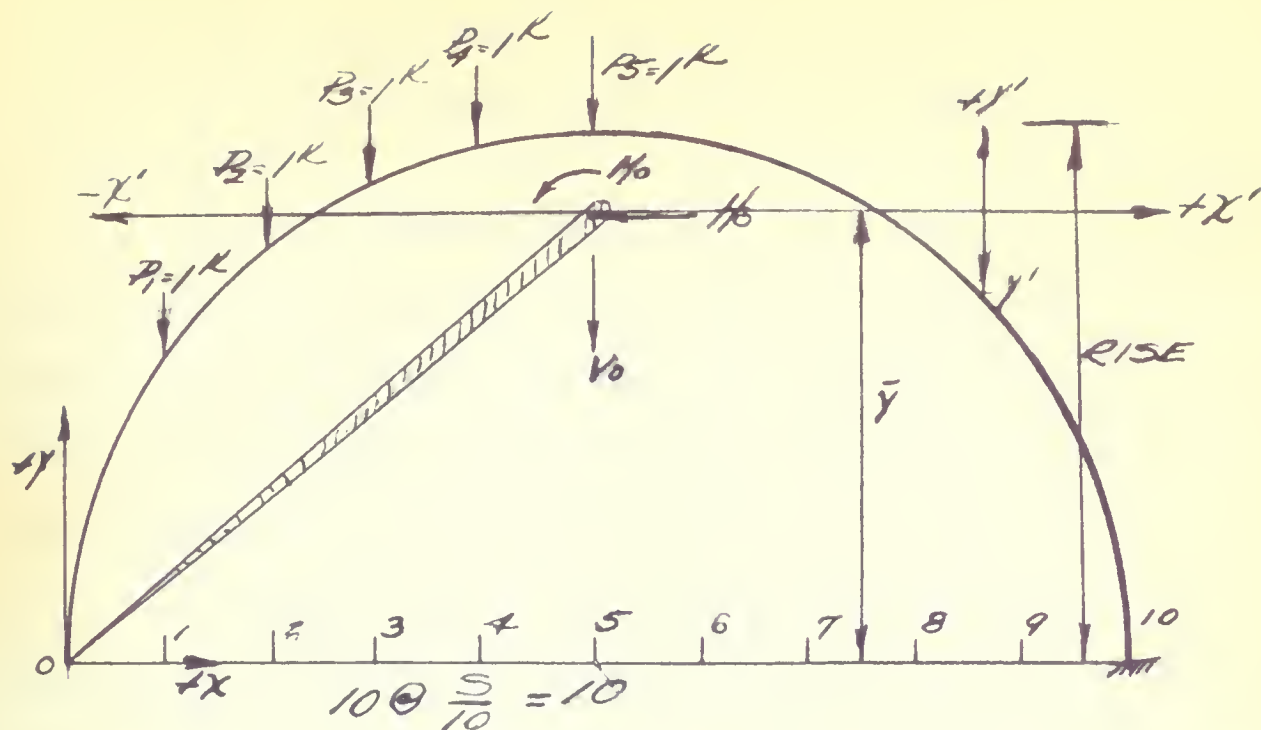
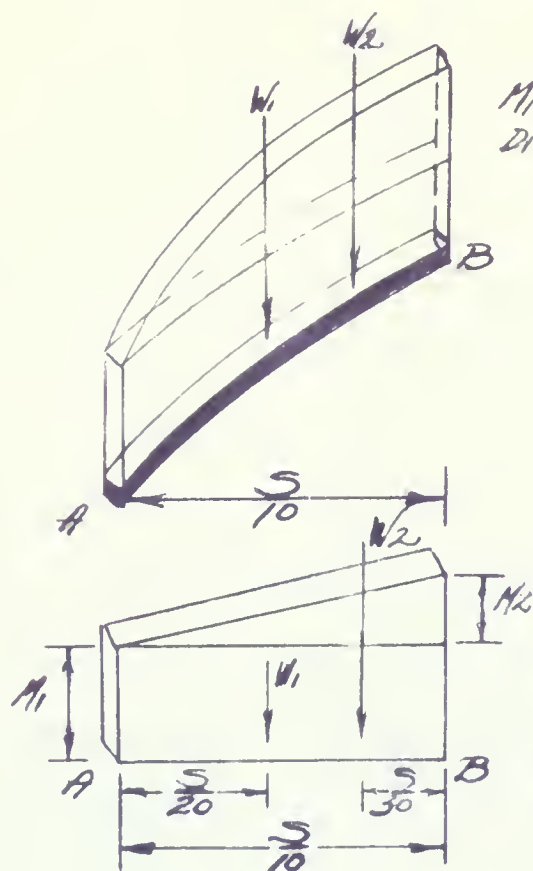


FIG. 1 PARABOLIC ARCH
 REDUNDANT REACTIONS APPLIED AT NEUTRAL POINT
 AND VERTICAL LOADS APPLIED AT $\frac{1}{10}$ SPAN POINTS.



M/EI LOADING ON AN ARC
 DIVIDED INTO RECTANGLE AND
 TRIANGLE SECTIONS

HORIZONTAL PROJECTION OF
 M/EI DIAGRAM LOADING ON
 THE ARC.
 W_1 - ELASTIC WT. OF RECT.
 W_2 - ELASTIC WT. OF TRIANGLE
 M_1 - MOMENT ON REAL STRUCTURE
 @ A
 $M_1 + M_2$ - MOMENT ON REAL
 STRUCTURE @ B

FIG. 2

right of the section tends to move up, with respect to the part on the right of the section, then the rotation of the corresponding section of the real structure is clockwise." ¹

After finding the elastic areas it is necessary to compute the neutral point, the centroid of the conjugate structure. Using the centroid so found, the values, ΔV , ΔH , and α , were found by the application at the neutral point of a unit horizontal load, a unit vertical load, and a unit moment.

By definition of the neutral point, a horizontal force at the neutral point causes only horizontal deflection of the neutral point, a unit moment only causes rotation, and a unit vertical force causes only vertical deflection. All deflections are in the same direction as the applied unit force or moment. (See Figure 1.)

Next, unit vertical loads were placed at each of the $1/10$ points across the span, and the deflections of the neutral point determined by means of the conjugate structure. These deflections were recorded as ΔV , ΔH , and θ .

In order to simplify the computations for the positions of the centroids of the loaded areas, the M/EI loading diagram on a given arc segment was divided into a triangle and rectangle. The total weights of the triangle and rectangle were considered to be located on the arc and at the $1/3$ point of the horizontal projection from the base of the triangle for the triangle and at the middle for the rectangle. (See Figure 2)

From the theories of the neutral point method, the following relationships hold true:

¹ "Indeterminate Structures" - J. S. Kinney, Sept., 1948

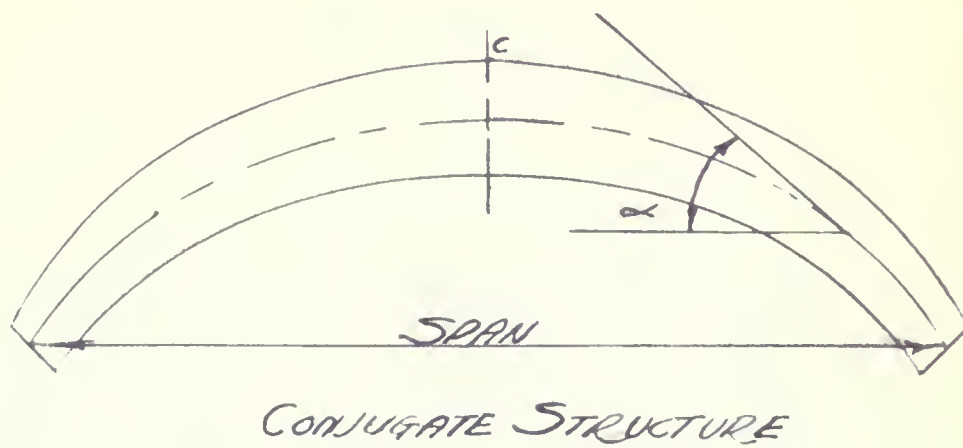
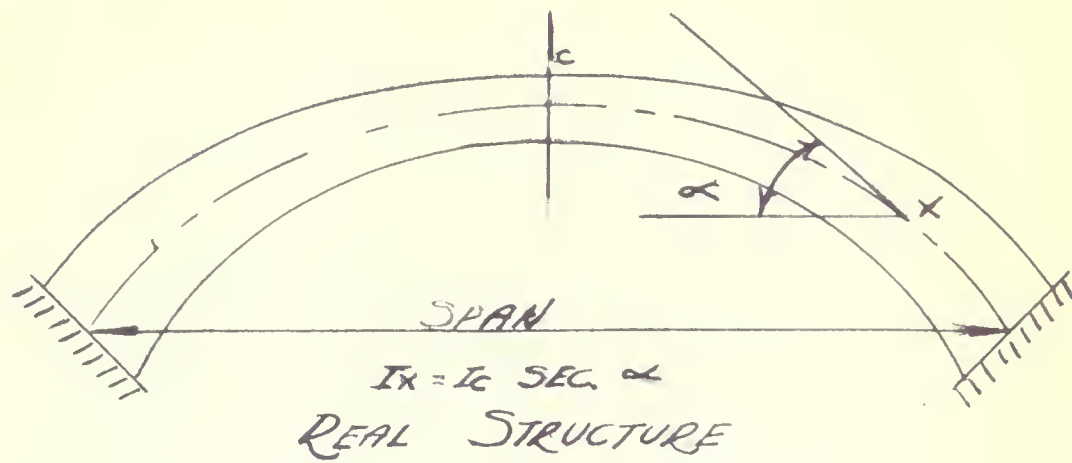


FIGURE 3

$$H_o = - \frac{\Delta H}{\Delta h} = H_L$$

$$V_o = - \frac{\Delta V}{\Delta v} = V_L$$

$$M_o = - \frac{\Delta \theta}{\Delta \alpha}$$

The shear, thrust, and moment at any point on the real structure can be determined by drawing a free body diagram of the arch with the right end rigidly fixed and the left end attached to the neutral point by means of the weightless bracket, and applying the loads on the structure and the reactions at the neutral point.

Using Figure 1 and applying the reactions at the neutral point,

$$M_L = M_o / H_o \bar{y} - V_o \frac{S}{2}$$

$$M_R = M_o / H_o \bar{y} / V_o \frac{S}{2} / P_x$$

with a positive moment indicating compression on the inside of the arch.

Sample computations will be found in Appendix D. It was found that the variations found by using the approximate method from the exact method were of a small magnitude.

Determination of Carry-over Factor

When an arch of varying moment of inertia is cut at the left springing, and redundant forces applied as shown in the Figure 4, deflection of the left springing will be as follows.

$$(1) \Delta V = M_A \Delta v_m / V_A \Delta v_v / H_A \Delta v_h$$

$$(2) \Delta H = M_A \Delta h_m / V_A \Delta h_v / H_A \Delta h_h$$

$$(3) \Delta \theta = M_A \Delta \alpha_m / V_A \Delta \alpha_v / H_A \Delta \alpha_h$$

Multiplying both sides of equations (1), (2), and (3) by EI_c results in the following.

$$(4) \text{ EICDELTA } V = M_A(\text{Eic delta vm}) / V_A(\text{Eic delta vv}) / H_A(\text{Eic delta vh})$$

$$(5) \text{ EICDELTA } H = M_A(\text{Eic delta hm}) / V_A(\text{Eic delta hv}) / H_A(\text{Eic delta hh})$$

$$(6) \text{ EICTHETA } = M_A(\text{Eic alpha mm}) / V_A(\text{Eic alpha mv}) / H_A(\text{Eic alpha mh})$$

However, to take in account the varying moment of inertia, it is necessary to use a term, "Relative $\frac{I_c}{I_x}$ ", to get the elastic area upon which the unit moment applied at the hinged end will act. The product of "Relative $\frac{1}{I_x}$ " times ds (arc distance) will be designated as "A" which is the elastic area of the conjugate structure. (Ic is considered as unity) In this particular case, it is assumed, although it is evident that this assumption is not entirely correct, that the Ix existing at the center of the arc segment considered, is the I existing over the entire segment.

It is therefore obvious by the use of virtual work that the following equations will give us the deflections. ($\delta = \text{SIGMA}$)

$$\text{Eic delta vm} = \text{Eic alpha mv} = \delta \frac{x ds}{I} = \delta Ax = \frac{\bar{S}}{2} \delta A$$

$$\text{Eic delta hm} = \text{Eic alpha mh} = \delta \frac{y ds}{I} = \delta Ay = \bar{y} \delta A$$

$$\text{Eic alpha mm} = \delta \frac{ds}{I} = \delta A$$

$$\text{Eic delta vh} = \text{Eic delta hv} = \delta xy \frac{ds}{I} = \delta xyA = \frac{\bar{S}\bar{y}}{2} \delta A$$

$$\text{Eic delta vv} = \delta x^2 \frac{ds}{I} / \delta I_y \text{ c.g.} = \delta Ax^2 / \delta I_y \text{ c.g.}$$

$$\text{Eic delta hh} = \delta y^2 \frac{ds}{I} / \delta I_x \text{ c.g.} = \delta Ay^2 / \delta I_x \text{ c.g.}$$

It is essential that the moments of inertia of the segments be added to Ax^2 and Ay^2 values as a 4% error will occur if this is not done.

Since the carry-over factor is numerically equal to the moment induced at the fixed end of a member when a unit moment produces rotation at the hinged end, there will be no horizontal or vertical

deflections. Consequently, ΔV and ΔH equals 0.

Solve equations (4) and (5) simultaneously and get V_A and H_A in terms of M_A . The carry-over factor can be found by setting up a free body as indicated in Figure 5 and solving for the induced moment at the fixed end.

When the induced moment at the fixed end is such that it operates in the opposite direction from the applied moment, the carry-over factor is considered negative.

It is obvious that any moment applied at the hinged end will produce a horizontal thrust. This thrust must be known in order to apply moment distribution to the solution of the arched bent. Every time a joint is rotated and the moments are balanced out at the joint a moment will be applied to the arch. Such applied moments will change the horizontal restraining forces at the ends of the arch.

Absolute Stiffness

The absolute stiffness of a member is the value of the moment necessary, applied at the simply supported end of a member arranged as a propped cantilever, to produce a rotation of one radian at the simply supported end. This definition states that ΔV and ΔH must again be equal to zero. Therefore, equations (4) and (5) are again equal to zero.

Solving these equations simultaneously will give the forces induced at the hinged end in terms of the applied moment. Substituting these values in equation (6) will give $EI\theta$ expressed in terms of the applied moment. Set θ equals 1 for unit rotation, and solve for M_A . This will give the absolute stiffness of the arch. After solving

several span lengths with the same rise/span ratio, it was determined that the absolute stiffness varies inversely as the span length and that the absolute stiffness remained constant for all rise/span ratios of the same span length. It is therefore concluded that the absolute stiffness is constant for any rise over span ratio and varies inversely as the span.

Spread

Any load placed on an arch causes a horizontal thrust at the ends of the arches. This thrust will cause spread in the arched bent and must be taken into consideration when solving the arched bent by moment distribution.

If a spread of ΔH feet occurs while there is no vertical deflection or rotation of the joint, reactions will be induced at the ends of the arch. These reactions will be determined by setting equations (4) and (6) equal to zero and solving equations (4), (5), and (6) simultaneously for H_A , V_A , and M_A in terms of $EI_c \Delta H$. After solving several span lengths for the same rise/span ratio, it was determined that V_A was equal to zero in all cases and that M_A varied inversely as the square of the span and that H_A varied inversely as the cube of the span. After solving the rise/span ratios, it was found that the induced moment varied directly as the quotient of the desired rise/span ratio over the rise/span ratio of 0.04. Integrating this knowledge, it was found that the moment varied inversely as the product of the span and the rise for that span. Similarly it was found that for the same span that the thrust varied as the square of the ratios of the rise/span ratios. Knowing this, it was found that the thrust varied inversely as the square of the rise and inversely as the span.

RESULTS

CONCLUSIONS

In this paper, tabulated data has been presented which will permit an engineer to solve any parabolic arched bent by moment distribution, provided that the moment of inertia of the arch varies as the secant of the angle alpha. Alpha is the angle between the tangent to the working line of the arch and the line at that point which is parallel to the springing line.

It was decided to omit all curves as the curves would not be large enough to permit accuracy which can be obtained by interpolation of the results.

It is believed that other arches should be solved with varying moments of inertia. This would undoubtedly save a great deal of time and energy.

As a result of this investigation, it was found that the following relationships are true.

- (1) Shear is constant for all spans.
- (2) Thrust varies directly as the length of span and inversely as the rise.
- (3) The value of the springing moments varies directly as the length of span, regardless of the rise.
- (4) The carry-over factor is constant for all spans and for all rise/span ratios.
- (5) The absolute stiffness varies inversely as the span.

RESULTS BASED ON A UNIT SPAN

Fixed-end Moments

Fixed-end moments are derived from Iselin and La Lande's Formulae given on page 5 and are based on a unit load.

<u>Load Point</u>	<u>M_R</u>	<u>M_L</u>	<u>V</u>	<u>H</u>
1	-.0113S	/.0605S	.9718	.0304 S/h
2	-.0320S	/.0640S	.8960	.0960 S/h
3	-.0473S	/.0367S	.7840	.1656 S/h
4	-.0480S	.0000	.6480	.2160 S/h
5	-.0317S	-.0317S	.5000	.2344 S/h

Note: Tension out is positive (/). The above data can be found exactly for any point from the formulae given on page 5.

Uniform Loading

$$\text{Thrust in kips} = 0.12 WS^2/h$$

$$\text{F.E.M. in ft. kips} = 0.0008 S^2W$$

$$W = \text{Load/lineal foot}$$

$$S = \text{Span in feet}$$

$$\text{Absolute Stiffness} = 9.068 EIc/S$$

$$\text{Carry-over Factor} = 0.3374$$

When joint A is free to rotate and a moment of M_A ft kips is applied at A, the following thrusts and shears are induced at A:

$$\text{Thrust in kips} = 0.833108 \frac{EIc M_A}{h}$$

$$\text{Shear in kips} = 0.66259 \frac{EIc M_A}{S}$$

When a spread, DELTA H, occurs, the following moments and thrust are developed at A:

$$\text{Thrust in kips} = \frac{11.2548 EIc \text{ DELTA } H}{Sh^2}$$

$$\text{Moment in ft. kips} = \frac{-7.5275 EIc \text{ DELTA } H}{Sh}$$

APPLICATION OF RESULTS

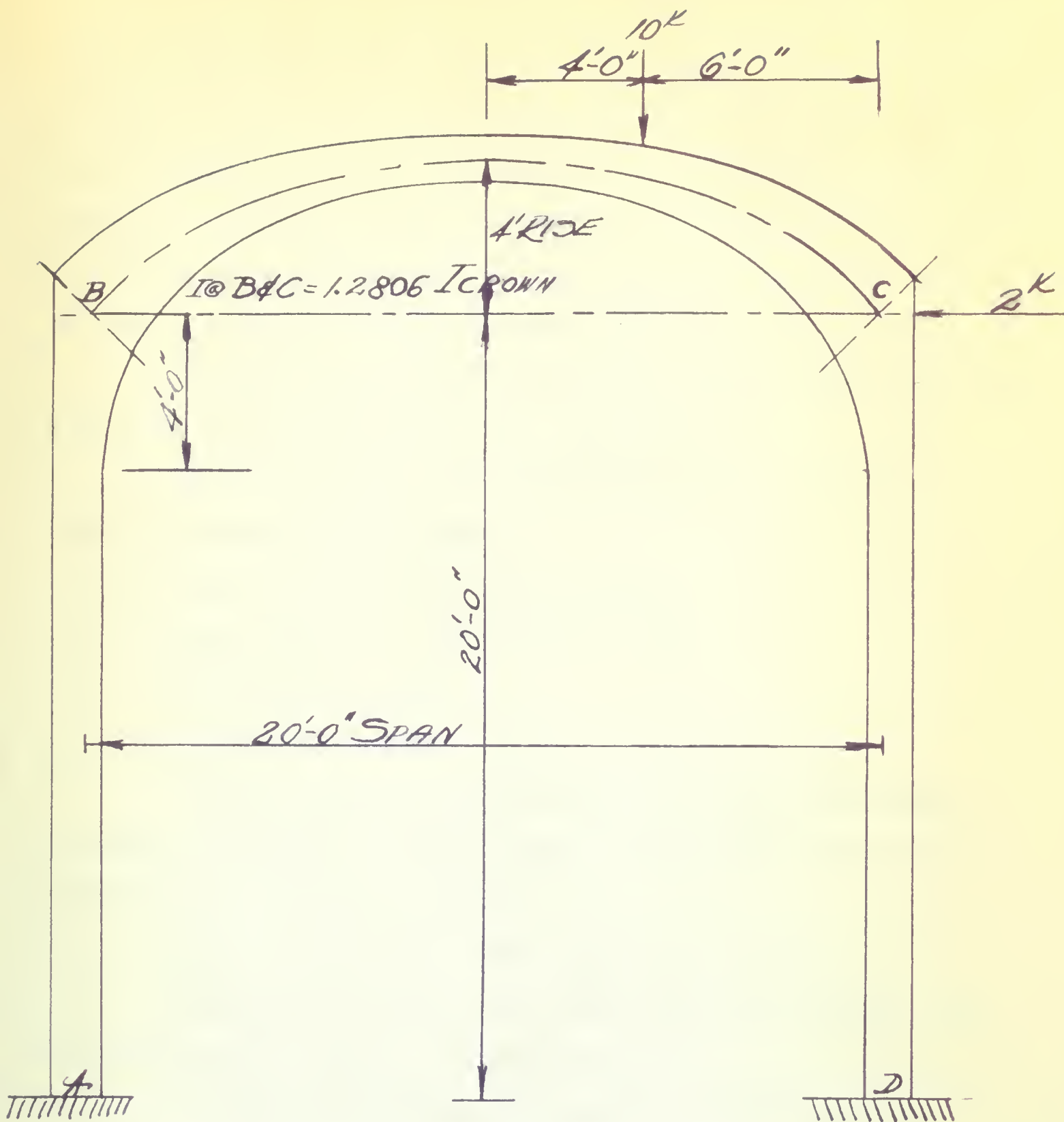


FIGURE 6

APPLICATION OF RESULTS TO THE SOLUTION OF AN ARCHED BENT
WITH VARYING MOMENT OF INERTIA BY MOMENT DISTRIBUTION

Given: An arched bent whose moment of inertia at any point is equal to $I_c \frac{ds}{dx}$ has a span of 20' 0" and a parabolic rise of 4' 0". The vertical legs are unsymmetric members with a parabolic haunch at the upper end which is 4' 0" long. The moment of inertia of the upper end of the haunch is equal to the moment of inertia at the end of the arch. The lower 16' 0" is equal to the moment of inertia at the center of the arch. (See Fig. 6.)

Determine the reactions at the supports and the moments at B and C which are caused by the given loads on the bent.

From the "Results", the following values for the fixed-end arched conditions can be determined.

$$M_{BC} = -.0473 \times 20 \times 10 = -9.46 \text{ ft. kips}$$

$$M_{CB} = .0367 \times 20 \times 10 = 7.34 \text{ ft. kips}$$

$$H_B = 8.3 \text{ kips} \quad \text{Right}$$

$$H_C = 8.3 \text{ kips} \quad \text{Left}$$

A positive moment indicates tension on the top of the member. In moment distribution, a positive moment is that moment which tends to rotate the joint clockwise. Therefore:

$$M_{BC} = -9.46 \text{ ft. kips} \quad \text{and} \quad M_{CB} = 7.34 \text{ ft. kips}$$

The absolute stiffness of the arch is equal to .4534 EI_c . The carry-over factor is -.337. (As noted on Pg. 14.)

The absolute stiffness of the parabolic haunched column is found from curves published in the "Theory and Practice of Reinforced Concrete" by C. W. Dunham.

	<u>A</u>	<u>B</u>
Absolute stiffness	.201 EI_c	.211 EI_c
Carry-over factor	.523	.498

Assume rotation of joints only and solve for the moments at all joints by moment distribution.

Joint	A		B			C		D
Member	AB	BA	BC			CB	CD	DC
K	.201	.211	.453			.453	.211	.201
K/ZK	--	.318	.682	Balancing		.682	.318	--
C.O.F.	.523	.498	-.337	Moment		-.337	.498	.523
F.E.M.			-9.46			-7.34		
	~~1.55~~	~~3.01~~	~~6.45~~	~~6.45~~		-2.17		
			-2.18		~~6.48~~	~~6.48~~	~~3.03~~	1.51
	~~0.35~~	~~0.69~~	~~1.49~~	~~1.49~~		-0.50		
			-0.12		~~0.34~~	~~0.34~~	~~0.16~~	0.08
	~~0.02~~	~~0.04~~	~~0.08~~	~~0.08~~		-0.03		
					~~0.02~~	~~0.02~~	~~0.01~~	
Mr	~~1.92~~	~~3.74~~	~~3.74~~			-3.20	~~3.20~~	~~1.59~~
Bal. Mom.				~~8.02~~	~~6.84~~			

A record of the balancing moment must be kept as this balancing moment changes the horizontal restraining force at the ends of the curved member.

For arched bents, the required horizontal restraining forces at the ends of the arch are made up of the following components:

- (1) The force necessary to restrain the horizontal thrust at the ends of the arch or caused by spread in the arch.
- (2) That force necessary to restrain the horizontal thrust at the ends of the arch caused by the balancing moments.
- (3) That necessary to balance the shears in the column members.
- (4) That force necessary to prevent sidesway when a horizontal load is applied to the bent.

A positive balancing moment at the left springing causes equal horizontal restraining forces at the two ends of the arch acting in a direction away from each other. A positive balancing moment at the right

springing causes equal horizontal restraining forces at the ends of the arch acting in the directions toward each other.

For the bent under consideration the horizontal restraining forces at the ends of the arch caused by the balancing moments is equal to $0.833108 \frac{HicMA}{h}$ which equals $0.20828 M_A$.

The artificial joint restraints (AJR) required to prevent the original bent from deflecting due to sidesway or spread are equal to the horizontal restraining forces at the ends of the arch.

<u>HRF caused by:</u>	<u>HRF_L</u>	<u>HRF_R</u>
1. Load on the arch	/8.30 kips	-8.30 kips
2. Balancing Moment at B (.20828)(8.02)= 1.67	-1.67 "	/1.67 "
3. Balancing Moment at C (.20828)(6.84)= 1.42	/1.42 "	-1.42 "
4. Shear in Member AB $\frac{1.92 + 3.74}{20}$	/0.28 "	
5. Shear in Member CD $\frac{1.59 + 3.20}{20}$		/0.24 "
6. 2 kip load at C	<u>/8.33 kips</u>	$\frac{+2.00}{-5.81} "$ kips

Note: To the right is positive (/)

Let x equal artificial joint restraint resistant against spread (AJRsp) and let y equal artificial joint restraint resisting sidesway (AJRes). Solve for x and y.

$$\begin{array}{rcl}
 x + y & = & +8.33 \text{ kips} \\
 x - y & = & 5.81 \text{ " } \\
 \hline
 2x & = & 14.14 \text{ kips} \\
 x & = & 7.07 \text{ " } \\
 y & = & 1.26 \text{ " }
 \end{array}$$

AJRsp = 7.07 kips acting from both ends inwards

AJRes = 1.26 kips to the right

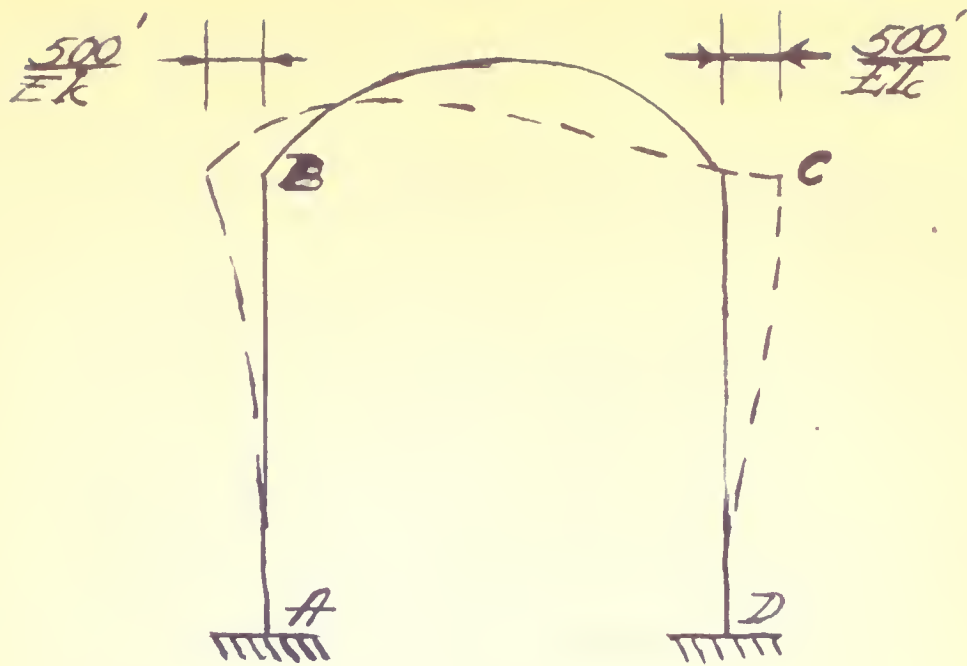


FIGURE 7

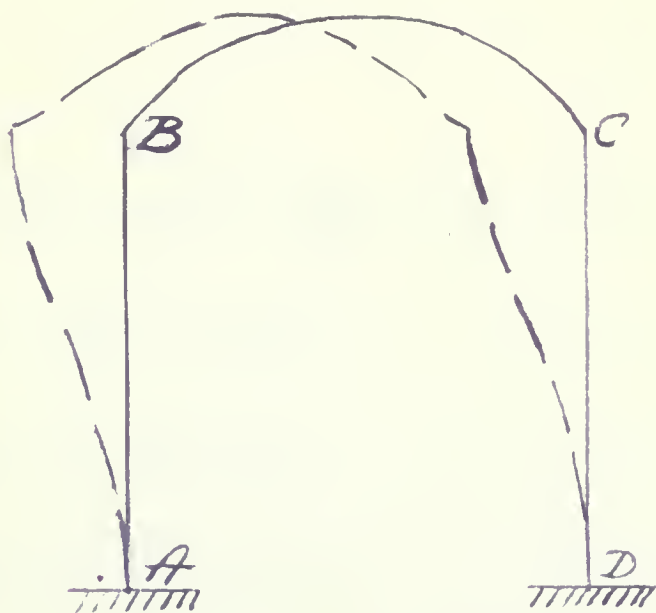


FIGURE 8

Assume a spread of 1000/EIc feet. (See Figure 7.) For a spread of the arched bent as shown, the following reactions are induced at the ends of the arch.

$$H_B = 11.2548 \frac{EIc \Delta H}{Sh^2} = \frac{11.2548 EIc \times 1000}{20 \times 16 \times EIc} = 35.17 \text{ kips left}$$

$$H_C = 35.17 \text{ kips right}$$

$$M_{BC} = 7.5275 \frac{EIc \Delta H}{Sh} = \frac{7.5275 EIc \times 1000}{20 \times 4 \times EIc} = 94.09 \text{ ft. kips}$$

$$M_{CB} = -94.09 \text{ ft. kips}$$

A positive moment indicates tension in the top of the arch.

For the moment distribution sign convention

$$M_{BC} = 94.09 \text{ ft. kips} \quad M_{CB} = -94.09 \text{ ft. kips}$$

The moments induced in the ends, B and C of the column members, are equal to $K_B (1 / C_B) \frac{\Delta H}{L}$

$$.211 EIc \frac{(1 / .498)}{20} \frac{(500)}{EIc} = 7.91 \text{ ft. kips}$$

where K_B = absolute stiffness at B and C_B = carry-over factor at B.

The moments induced at end A and D of the column members are equal to $K_A (1 / C_A) \frac{\Delta H}{L}$

$$.201 EIc \frac{(1 / .523)}{20} \frac{(500)}{EIc} = 7.67 \text{ ft. kips}$$

$$M_{AB} = -7.67 \text{ ft. kips}$$

$$M_{BA} = -7.91 \text{ ft. kips}$$

$$M_{CD} = 7.91 \text{ ft. kips}$$

$$M_{DC} = 7.67 \text{ ft. kips}$$

Solve for the moments at the joints caused by the assumed spread.

Joint	A			B			C		D
Member	AB	BA	BC		CB	CD	DC		
K	.201	.211	.453		.453	.211	.201		
K/ΣK	--	.318	.682	Balancing	.682	.318	--		
C.O.F.	.523	.498	-.337	Moment	-.337	.498	.523		
F.E.M.	-7.67	-7.91	94.09		-94.09	7.91	7.67		
	-13.65	-27.41	-58.77	-58.77	19.81				
		-15.25		45.26	45.26	21.11	10.52		
	2.42	4.85	10.40	10.40	-3.50				
		-0.81		2.39	2.39	1.11	0.55		
	0.13	0.26	0.55	0.55	-0.19				
		-0.04		0.13	0.13	0.06	0.03		
		0.01	0.03	0.03	-0.01				
				0.01	0.01				
Msp	-18.77	-30.20	30.20		-30.19	30.19	18.77		
Bal. Moment			-47.79	47.79					

The consistant joint force (CJF) required at B and C to maintain the assumed spread is equal to the horizontal restraining force (HRF) at the ends of the arc.

HRF caused by:	HRF _L	HRF _R
1. Spread in the arch	-35.17 kips	35.17 kips
2. Balancing Moment at B (.20829)(47.79)= 9.94	9.94 "	-9.94 "
3. Balancing Moment at C (.20829)(47.79)= 9.94	9.94 "	-9.94 "
4. Shear in Member AB $\frac{18.77 - 30.20}{20}$	-2.45 "	
5. Shear in Member CD		$\frac{2.45}{17.74}$ "
CJFsp	-17.74 "	17.74 "

Note: To the right is positive (✓)

It is thus seen that a CJFsp of 17.73 kips at each end of the arch acting away from each other will produce the given result and since an AJRsp of 7.07 kips was required to restrain the original structure from spreading, the actual spread will cause the moments at the various joints to change by an amount equal to

$$\frac{AJRsp}{CJFsp} \times Msp = \frac{7.07}{17.74} \times Msp = 0.398 Msp$$

Next assume a sideways of 1000/EIc feet to the left. (See Figure 8).

The moments induced in the ends of the column members equals:

$$K (1/c) \frac{\Delta H}{L}$$

$$M_{AB} = -15.34 \text{ ft. kips}$$

$$M_{BA} = -15.82 \text{ ft. kips}$$

$$M_{CD} = -15.82 \text{ ft. kips}$$

$$M_{DC} = -15.34 \text{ ft. kips}$$

Solve for moments at the joints caused by sideways.

Joint	A			B			C		D
Member	AB	BA	BC	CB	CD	DC			
K	.201	.211	.453	.453	.211	.201			
K/EK	--	.318	.682	.682	.318	--			
C.O.F.	.523	.498	-.337	-.337	.498	.523			
F.E.M.	-15.34	-15.82			-15.82	-15.34			
	~~2.51~~	~~5.03~~	~~10.79~~	~~10.79~~	~~3.60~~	~~3.08~~			
			-4.47	13.24	13.24	6.18			
	~~0.71~~	~~1.42~~	~~3.05~~	~~3.05~~	~~1.03~~	~~0.15~~			
			-0.24	0.70	0.70	0.33			
	~~0.04~~	~~0.08~~	~~0.16~~	~~0.16~~	~~0.05~~	~~0.01~~			
				0.04	0.04	0.01			
Mss	-12.09	-9.29	9.29	9.30	-9.30	-12.11			
Bal. Mom.			14.00	13.98					

The consistant joint forces, CJF, required at B and C to maintain the given sideways are equal to the horizontal restraining forces at C and B and are equal to the following:

<u>HRF caused by:</u>	<u>HRF_L</u>	<u>HRF_R</u>
1. Balancing moment at joint B (.20829)(14.00) = 2.92	-2.92 kips	/2.92 kips
2. Balancing moment at joint C	/2.92 "	-2.92 "
3. Shear in member AB $\frac{12.09 - 9.29}{20} = 1.07$	-1.07 "	
4. Shear in member CD CJFss	<u>-1.07 kips</u>	<u>-1.07 kips</u>
		<u>-1.07 "</u>

Note: To the right is positive (/).

Thus a consistent joint force of 1.07 kips at each end of the arch acting towards the left will produce the given sidesway; and since an AJRss of 1.26 kips acting towards the right was required to keep the original structure from swaying, the actual sidesway will cause the moments at various points to change by an amount equal to:

$$\frac{AJRss}{CJFss} \times Mss = \frac{1.26}{1.07} \times Mss = 1.177 Mss$$

Factor	<u>M_{AB}</u>	<u>M_{BA}</u>	<u>M_{BC}</u>	<u>M_{CB}</u>	<u>M_{CD}</u>	<u>M_{DC}</u>
M _R	/1.92	/ 3.74	- 3.74	- 3.20	/ 3.20	/ 1.59
0.398 Msp	- 7.47	-12.00	/12.00	-12.00	/12.00	/7.47
1.177 Mss	-14.25	-10.95	/10.95	/10.95	-10.95	-14.25
Final Mom.	-19.80	-19.21	/19.21	- 4.25	/ 4.25	- 5.19
Check by Conj.Struct.	-19.70	-19.40	/19.40	- 4.40	/ 4.40	- 5.20

$$H_A = \frac{19.80 - 19.21}{20} = \frac{39.01}{20} = 1.95 \text{ kips right}$$

$$H_D = \frac{4.25 - 5.19}{20} = \frac{9.44}{20} = .04726 \text{ kips right}$$

$$V_A = \frac{60 - 14.96}{20} = \frac{74.96}{20} = 3.748 \text{ kips up}$$

$$V_D = \frac{140 - 14.96}{20} = \frac{125.04}{20} = 6.252 \text{ kips up}$$

APPENDIX A

TABULATION OF ARC LENGTHS

Tabulation of Arc Lengths

Rise/span 0.04

<u>Segment</u>	<u>20' Span</u>	<u>30' Span</u>
0-1	2.02066	3.03099
1-2	2.01262	3.01894
2-3	2.00648	3.00972
3-4	2.00240	3.00358
4-5	2.00034	3.00051

Rise/span 0.08

0-1	2.08112	3.12169
1-2	2.05037	3.07556
2-3	2.02607	3.03911
3-4	2.00927	3.01385
4-5	2.00133	3.00205

Rise/span 0.12

0-1	2.17847	3.26770
1-2	2.11035	3.16553
2-3	2.05747	3.08621
3-4	2.02137	3.03206
4-5	2.00307	3.00460

Rise/span 0.20

0-1	2.46094	3.69140
1-2	2.29227	3.43840
2-3	2.15553	3.23329
3-4	2.05874	3.08810
4-5	2.00850	3.01275

Rise/span 0.30

0-1	2.94527	4.41791
1-2	2.61420	3.92130
2-3	2.33538	3.50307
3-4	2.12966	3.19449
4-5	2.01902	3.02853

Rise/span 0.40

0-1	3.50794	5.26190
1-2	3.00545	4.50817
2-3	2.56533	3.84800
3-4	2.22470	3.33704
4-5	2.03365	3.05047

APPENDIX B

MOMENT OF INERTIA I_x AND RELATIVE $1/I_x$

Mathematical Derivation of I_x

$$(1) I_x = I_c \sec \alpha = I_c \frac{ds}{dx}$$

The equation for the parabola is as follows:

$$y = ax^2 \quad \text{where } a = 4h/L^2$$

From trigonometry, it is seen that for infinitesimal lengths the following is true:

$$(2) ds = (dy^2 + dx^2)^{.5}$$

Therefore, we can replace dy^2 by taking the first derivative of the equation for the parabola and squaring the term.

$$y = ax^2 \quad dy = 2ax \, dx$$

$$(3) dy^2 = 4a^2 x^2 \, dx^2$$

Substituting for dy^2 in formula (2) we get,

$$ds = (4a^2 x^2 \, dx^2 + dx^2)^{.5}$$

$$ds = dx (4a^2 x^2 + 1)^{.5}$$

$$\frac{ds}{dx} = (4a^2 x^2 + 1)^{.5}$$

Substituting for $\frac{ds}{dx}$ in equation (1) the following results:

$$I_x = I_c (4a^2 x^2 + 1)^{.5}$$

We are assuming I_c equal to unit, therefore, the equation becomes the following:

$$I_x = (4a^2 x^2 + 1)^{.5}$$

Determination of Relative Ix and Relative 1/Ix

20' span, rise/span = .08

$$a = \frac{4(1.6)}{400} = .016$$

$\frac{x}{9}$	$\frac{x^2}{81}$	$\frac{a^2}{.000256}$	$\frac{4a^2x^2}{.082944}$	$\frac{4a^2x^2}{1.08294}$	$\frac{(0)^5}{1.0406}$	$\frac{1/I}{.9610}$
9	81	.000256	.082944	1.08294	1.0406	.9610
7	49	.000256	.050176	1.050176	1.0247	.9759
5	25	.000256	.02560	1.02560	1.0127	.9875
3	9	.000256	.009216	1.009216	1.0046	.9954
1	1	.000256	.001024	1.001024	1.0005	.9995

20' span, rise/span = .04

$$a = \frac{4(.8)}{400} = .008$$

9	81	.000064	.020736	1.020736	1.0103	.9898
7	49	.000064	.012544	1.012544	1.0062	.9938
5	25	.000064	.0064	1.0064	1.0032	.9968
3	9	.000064	.002304	1.002304	1.0012	.9988
1	1	.000064	.000256	1.000256	1.0001	.9999

30' span, rise/span = .04

$$a = \frac{4(1.2)}{900} = .00533$$

13.5	182.25	.0000284	.020733	1.020733	1.0103	.9898
10.5	110.25	.0000284	.012542	1.012542	1.0062	.9938
7.5	56.25	.0000284	.006399	1.006399	1.0032	.9968
4.5	20.25	.0000284	.0023036	1.002304	1.0012	.9988
1.5	2.25	.0000284	.0002560	1.000256	1.0001	.9999

20' span, rise/span = .40

$$a = \frac{4n}{L^2} = \frac{4(8)}{400} = .08$$

9	81	.0064	2.0736	3.0736	1.7532	.5704
7	49	.0064	1.2544	2.2544	1.5015	.6660
5	25	.0064	0.6400	1.6400	1.2806	.7809
3	9	.0064	0.2304	1.2304	1.1092	.9016
1	1	.0064	0.0256	1.0256	1.0127	.9875

20' span, rise/span = .30

$$a = \frac{4(6)}{400} = .06$$

9	81	.0036	1.1664	2.1664	1.4719	.6794
7	49	.0036	0.7056	1.7056	1.3060	.7657
5	25	.0036	0.3600	1.3600	1.1662	.8575
3	9	.0036	0.1296	1.1296	1.0628	.9409
1	1	.0036	0.0144	1.0144	1.0072	.9929

20' span, rise/span .20

$$a = \frac{4(4)}{400} = .04$$

9	81	.0016	.5184	1.5184	1.2322	.8116
7	49	.0016	.3136	1.3136	1.1461	.8725
5	25	.0016	.1600	1.1600	1.0770	.9285
3	9	.0016	.0576	1.0576	1.0284	.9724
1	1	.0016	.0064	1.0064	1.0032	.9968

APPENDIX C

TABULATION OF ELASTIC AREAS AND
CENTROID (NEUTRAL POINT) OF ARCH

Tabulation of Elastic Areas

Rise/span .04 Span 20'	<u>Segment</u>	<u>Arc Length</u>	<u>Relative 1/I</u>	<u>A</u>
	0-1	2.0207	.9898	2.0001
	1-2	2.0126	.9938	2.0001
	2-3	2.0065	.9968	2.0001
	3-4	2.0024	.9988	2.0000
	4-5	2.0003	.9999	2.0001
	0-10			20.0008
Rise/span .04 Span 30'	0-1	3.0310	.9898	3.0001
	1-2	3.0189	.9938	3.0002
	2-3	3.0097	.9968	3.0001
	3-4	3.0036	.9988	3.0000
	4-5	3.0005	.9999	3.0002
	0-10			30.0012
Rise/span .08 Span 20'	0-1	2.0811	.9610	1.9999
	1-2	2.0504	.9759	2.0010
	2-3	2.0261	.9875	2.0008
	3-4	2.0093	.9954	2.0001
	4-5	2.0013	.9995	2.0003
	0-10			20.0042
Rise/span .20 Span 20'	0-1	2.4609	.8116	1.9973
	1-2	2.2923	.8725	2.0000
	2-3	2.1555	.9285	2.0014
	3-4	2.0587	.9724	2.0019
	4-5	2.0085	.9968	2.0021
	0-10			20.0054

Rise/span .30 Span 20'	<u>Segment</u>	<u>Arc Length</u>	<u>Relative l/I</u>	<u>A</u>
	0-1	2.9453	.6794	2.0010
	1-2	2.6142	.7657	2.0017
	2-3	2.3354	.8575	2.0026
	3-4	2.1297	.9409	2.0038
	4-5	2.0190	.9929	2.0046
	0-10			20.0274
Rise/span .40 Span 20'	0-1	3.5079	.5704	2.0009
	1-2	3.0054	.6660	2.0016
	2-3	2.5653	.7809	2.0032
	3-4	2.2247	.9016	2.0058
	4-5	2.0336	.9875	2.0082
	0-10			20.0394

Neutral Point of Arch - $\bar{y}$

<u>Rise/span</u>	<u>Span</u>	<u>$\bar{y}$</u>
.04	20	0.536
.08	20	1.072
.20	20	2.681
.30	20	4.0211
.40	20	5.363
.04	30	0.804
.08	30	1.608
.20	30	4.022
.30	30	6.0316
.40	30	8.0440

APPENDIX D

SOLUTION OF ARCHED BENT FOR M_L , M_R , H , and V

1 Kip Vertical at Point 1

0	1	2	3	4	5	6	7	8	9	10	11	
Sect.	A	RECTANGLE				TRIANGLE				THETA 3x1x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8
		M/EI	y'(dH)	x'(dV)	THETA 1x2	Ht.	y'(dH)	x'(dV)	THETA 3x1x6			
1-2	2.0001	-	-	-	-	2	-.0916	-6 2/3	2.0001	-0.1832	-13.3340	
2-3	2.0001	2	f.064	-5	4.0002	2	f.0898	-4 2/3	2.0001	f0.2560 f0.1796	-20.0010 -9.3337	
3-4	2.0000	4	f.192	-3	8.0000	2	f.2071	-2 2/3	2.0000	f1.5360 f0.4142	-24.0000 -5.3333	
4-5	2.0001	6	f.256	-1	12.0006	2	f.2604	-2/3	2.0001	f3.0722 f0.5208	-12.0006 -1.3334	
5-6	2.0001	8	f.256	f1	16.0008	2	f.2498	f1 1/3	2.0001	f4.0962 f0.4996	f16.0008 f2.6667	
6-7	2.0000	10	f.192	f3	20.0000	2	f.1751	f3 1/3	2.0000	f3.8400 f0.3502	f60.0000 f6.6667	
7-8	2.0001	12	f.064	f5	24.0012	2	f.0364	f5 1/3	2.0001	f1.5361 f0.0728	f120.0060 f10.6671	
8-9	2.0001	14	-.128	f7	28.0014	2	-.1662	f7 1/3	2.0001	-3.5842 -0.3324	f196.0098 f14.6673	
9-10	2.0001	16	-.384	f9	32.0016	2	-.4329	f9 1/3	2.0001	-12.2886 -0.8658	f288.0144 f18.6675	
										<u>18.0007</u> -0.8805		
										THETA = 144.0058 f 18.0007 = 162.0065		

Rise - 0.8', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

1 Kip Vertical at Point 2

0	1	2	M/EI RECTANGLE			5	6	M/EI TRIANGLE		9	10	11
Sect.	A	Ht.	y'(dH)	x'(dV)	THETA 1x2	Ht.	y'(dH)	x'(dV)	THETA 1/2x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8	
2-3	2.0001	-	-	-	-	2	f.0898	-4 2/3	2.0001	f0.1796	-9.3337	
3-4	2.0000	2	f.192	-3	4.0000	2	f.2071	-2 2/3	2.0000	f0.7680 f0.4142	-12.0000 -5.3333	
4-5	2.0001	4	f.256	-1	8.0004	2	f.2604	-2/3	2.0001	f2.0481 f.5208	-8.0004 -1.3334	
5-6	2.0001	6	f.256	f1	12.0006	2	f.2498	f1 2/3	2.0001	f3.0722 f.4996	f12.0006 f2.6667	
6-7	2.0000	8	f.192	f3	16.0000	2	f.1751	f3 1/3	2.0000	f3.0720 f0.3502	f48.0000 f6.6669	
7-8	2.0001	10	f.064	f5	20.0010	2	f.0364	f5 1/3	2.0001	f1.2801 f.0728	f100.0050 f10.6671	
8-9	2.0001	12	-.128	f7	24.0012	2	-.1662	f7 1/3	2.0001	-3.0722 -.3324	f168.0084 f14.6673	
9-10	2.0001	14	-.384	f9	28.0014	2	-.4329	f9 1/3	2.0001	-10.7525 -.8658	f252.0126 f18.6675	
					<u>112.0046</u>				<u>16.0006</u>	<u>-2.7453</u>	<u>f597.3613</u>	
					THETA = 112.0046 f 16.0006 = 128.0052							

Rise - 0.8', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

1 Kip Vertical at Point 3

0	1	2	3	4	5	6	7	8	9	10	11
Sect.	A	Ht.	M/EI	RECTANGLE	THETA 1x2	Ht.	M/EI	TRIANGLE	THETA 1/2x1x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8
			y'(dH)	x'(dV)			y'(dH)	x'(dV)			
3-4	2.0000	-	-	-	-	2	f.2071	-2 2/3	2.0000	f0.4142	-5.3333
4-5	2.0001	2	f.256	-1	4.0002	2	f.2604	-2/3	2.0001	f1.0240 f0.5208	-4.0002 -1.3335
5-6	2.0001	4	f.256	f1	6.0004	2	f.2498	f1 1/3	2.0001	f2.0480 f0.4996	f8.0004 f2.6668
6-7	2.0000	6	f.192	f3	12.0006	2	f.1751	f3 1/3	2.0000	f2.3041 f.3502	f36.0018 f6.6667
7-8	2.0001	8	f.064	f5	16.0008	2	f.0364	f5 1/3	2.0001	f1.0241 f.0728	f80.0040 10.6672
8-9	2.0001	10	-.128	f7	20.0010	2	-.1662	f7 1/3	2.0001	-2.5601 -.3324	f140.0070 f14.6674
9-10	2.0001	12	-.384	f9	24.0012	2	-.4329	f9 1/3	2.0001	-9.2164 -.8658	f216.0108 f18.6676
					<u>84.0042</u>				<u>14.0005</u>	<u>-4.7169</u>	<u>f522.6897</u>
					THETA = 84.0042 f 14.0005 = 98.0047						

Rise - 0.6', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

1 Kip Vertical at Point 4

0	1	2	3	4	5	6	7	8	9	10	11
Sect.	A	Ht.	M/ET y'(dH)	RECTANGLE x'(dV)	THETA 1x2	Ht.	M/ET y'(dH)	TRIANGLE x'(dV)	THETA 1/2x1x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8
4-5	2.0001	-	-	-	-	2	~~f~~ .2604	-2/3	2.0001	~~f~~ 0.5208	-1.3335
5-6	2.0001	~~f~~ 2	~~f~~ .2560	~~f~~ 1	4.0002	2	~~f~~ .2498	~~f~~ 1 1/3	2.0001	~~f~~ 1.0240 ~~f~~ 0.4996	~~f~~ 4.0002 ~~f~~ 2.6668
6-7	2.0000	~~f~~ 4	~~f~~ 1920	~~f~~ 3	8.0000	2	~~f~~ .1751	~~f~~ 3 1/3	2.0000	~~f~~ 1.5360 ~~f~~ 0.3502	~~f~~ 24.0000 ~~f~~ 6.6667
7-8	2.0001	~~f~~ 6	~~f~~ .0640	~~f~~ 5	12.0006	2	~~f~~ .0364	~~f~~ 5 1/3	2.0001	~~f~~ 0.7680 ~~f~~ 0.0728	~~f~~ 60.0030 ~~f~~ 10.6672
8-9	2.0001	~~f~~ 8	- .1280	~~f~~ 7	16.0008	2	- .1662	~~f~~ 7 1/3	2.0001	-2.0180 -0.3324	~~f~~ 112.0042 ~~f~~ 14.6674
9-10	2.0001	~~f~~ 10	- .3840	~~f~~ 9	20.0010	2	- .4329	~~f~~ 9 1/3	2.0001	-7.6803 -0.8658	~~f~~ 180.0090 ~~f~~ 18.6676
					<u>60.0026</u> THETA = 60.0026 ~~f~~ 12.0005 = 72.0031				<u>12.0005</u>	<u>-6.1341</u>	<u>~~f~~432.0142</u>

Rise - 0.8', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

1 Kip Vertical at Point 5

Sect.	A	M/EI RECTANGLE					M/EI TRIANGLE					DELTA H	DELTA V
		0	1	2	3	4	5	6	7	8	9		
		Ht.	y' (dH)	x' (dV)	THETA		Ht.	y' (dH)	x' (dV)	THETA			
						1x2					3x1x6	5x3 9x7	5x4 9x8
5-6	2.0001	-	-	-	-		f2	f.2498	f1.3333	2.0001	f0.4996	f2.6668	
6-7	2.0000	f2	f.1920	f3	4.0000		f2	f.1751	f3.3333	2.0000	f0.7680 f0.3502	f12.0000 f6.6667	
7-8	2.0001	f4	f.0640	f5	8.0004		f2	f.0364	f5.3333	2.0001	f0.5120 f0.0728	f40.0020 f10.6672	
8-9	2.0001	f6	- .1280	f7	12.0006		f2	- .1662	f7.3333	2.0001	-1.5360 -0.3324	f84.0042 f14.6674	
9-10	2.0001	f8	- .3840	f9	16.0008		f2	- .4329	f9.3333	2.0001	-6.1442 -0.8658	f144.0072 f18.6676	
					40.0018					10.0004	-6.6758	f333.3491	

THETA = 40.0018 / 10.0004 = 50.0022

Rise - 0.8'
Span - 20'
y - 0.536'
Rise/span 0.04

1 Kip Vertical at Point 6

0	1	2	3	4	5	6	7	8	9	10	11
Sect.	A	Ht.	M/EI y'(dH)	RECT ANGLE x'(dV)	THETA 1x2	Ht.	M/EI y'(dH)	TRI ANGLE x'(dV)	THETA 1/2x1x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8
6-7	2.0000	-	-	-	-	f2	f1.1751	f3 1/3	2.0000	f3.3502	f6.6667
7-8	2.0001	f2	f0.0640	f5	4.0002	f2	f0.0364	f5 1/3	2.0001	f2.2560 f0.0728	f20.0010 f10.6672
8-9	2.0001	f4	-0.1280	f7	8.0004	f2	-0.1662	f7 1/3	2.0001	-1.0241 -0.3324	f56.0028 f14.6674
9-10	2.0001	f6	-0.3840	f9	12.0006	f2	-0.4329	f9 1/3	2.0001	-4.6082 -0.8658	f108.0054 f18.6676
					<u>24.0012</u>				<u>8.0006</u>	<u>-6.1515</u>	<u>f234.6781</u>

THETA = 24.0012 / 8.0006 = 32.0018

Rise - 0.8', Span - 20', Rise/span - 0.04, y - 0.536'

1 Kip Vertical at Point 7

0	1	2	3	4	5	6	7	8	9	10	11
Sect.	A	Ht.	RECTANGLE			Ht.	M/EI TRIANGLE		THETA $\frac{1}{2}x^2$	DELTA H $5x^3$ $9x^7$	DELTA V $5x^4$ $9x^8$
			$y'(dH)$	$x'(dV)$	THETA $1x^2$		$y'(dH)$	$x'(dV)$			
7-8	2.0001	-	-	-	-	f2	f.0364	f5 1/3	2.0001	f.0728	f10.6672
8-9	2.0001	f2	-.1280	f7	4.0002	f2	-.1662	f7 1/3	2.0001	-.5120 -.3324	f28.0014 f14.6674
9-10	2.0001	f4	-.3840	f9	8.0004	f2	-.4329	f9 1/3	2.0001	-3.0720 -0.8658 -4.7094	f72.0036 f18.6676 f144.0072
					<u>12.0006</u>				<u>6.0003</u>		
					THETA = 12.0006						

1 Kip Vertical at Point 8

8-9	2.0001	-	-	-	-	f2	-.1662	f7 1/3	2.0001	-0.3324	f14.6674
9-10	2.0001	f2	-.3840	f9	4.0002	f2	-.4329	f9 1/3	2.0001	-1.5360	f36.0018
					<u>4.0002</u>				<u>4.0002</u>	<u>-0.8658</u>	<u>f18.6676</u>
					THETA = 4.0002					<u>-2.7342</u>	<u>f69.3368</u>

1 Kip Vertical at Point 9

9-10	2.0001	-	-	-	-	f2	-.4329	f9 1/3	2.0001	-.8658	f18.6675
									<u>2.0001</u>	<u>-2.7342</u>	<u>f69.3368</u>

Rise 0.8', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

1 Kip Vertical at Neutral Point

0	1	2	3	4	5	6	7	8	9	10	11
Sect.	A	Ht.	M/EI y'(dH)	RECTANGLE x'(dV)	ANGLE THETA 1x2	Ht.	M/EI y'(dH)	TRIANGLE x'(dV)	THETA 1/2x1x6	DELTA H 5x3 9x7	DELTA V 5x4 9x8
0-1	2.0001	-8	-0.3840	-9	-16.0008	-2	-0.4329	-9.3333	-2.0001		144.0072 18.6676
1-2	2.0001	-6	-0.1280	-7	-12.0006	-2	-0.1662	-7.3333	-2.0001		84.0042 14.6674
2-3	2.0001	-4	0.0648	-5	-8.0004	-2	0.0364	-5.3333	-2.0001		40.0020 10.6672
3-4	2.0000	-2	0.1920	-3	-4.0000	-2	0.1751	-3.3333	-2.0000		12.0000 6.6667
4-5	2.0001	-	0.2560	-1	-	-2	0.2498	-1.3333	-2.0001		2.6668
5-6	2.0001	-	0.2560	1	-	2	0.2498	1.3333	2.0001		2.6668
6-7	2.0000	2	0.1920	3	4.0000	2	0.1751	3.3333	2.0000		12.0000 6.6667
7-8	2.0001	4	0.0648	5	8.0004	2	0.0364	5.3333	2.0001		40.0020 10.6672
8-9	2.0001	6	-0.1280	7	12.0006	2	-0.1662	7.3333	2.0001		84.0042 14.6674
9-10	2.0001	8	-0.3840	9	16.0008	2	-0.4329	9.3333	2.0001		144.0072 18.6676
										delta vv = 666.6982	

Rise - 0.6', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

1 Kip Horizontal at Neutral Point

0	1	2	3	4	5	6	7	8	9	10	11
Sect.	A	M/EI RECTANGLE		M/EI TRIANGLE		Ht. y'(dH)		x'(dV)		DELTA H	DELTA V
		Ht.	y'(dH)	x'(dV)	THETA 1x2		y'(dH)	x'(dV)	THETA 1/2x1x6	5x3 9x7	5x4 9x8
0-1	2.0001	f.248	-.384	-9	f.4960	f.288	-.4329	-9.3333	f.288	-.190464 -.124675	
1-2	2.0000	f.024	-.128	-7	f.048	f.224	-.1662	-7.3333	f.224	-.006144 -.037229	
2-3	2.0001	-	-	-5	-	f.024 -.136	-.0147 f.0968	-5.903 -4.572	f.024 -.136	-.000353 -.0131648	
3-4	2.0001	-.136	f.192	-3	-.272	-.0960	f.2071	-2.6667	-.096	-.052224 -.0198816	
4-5	2.0001	-.232	f.256	-1	-.464	-.0320	f.2611	-0.6667	-.032	-.118784 -.0083552	
						delta hh = 2 x -.571275 = -1.14255				-.571275	

Rise - 0.8', Span - 20', Rise/span - 0.04, $\bar{y}$ - 0.536'

Load at Point	<u>H_o</u>	<u>V_o</u>	<u>M_o</u>	<u>$\bar{y}$ H_o</u>	<u>10 V_o</u>	<u>M_L</u>	<u>M_R</u>	1 Foot Span	
								<u>M_L</u>	<u>M_R</u>
1	- .77065	.97200	-8.1000	- .41307	-9.7200	~~1~~ 1.2069	-0.2264	~~1~~ 0.0603	-0.0113
2	-2.40278	.89600	-6.4000	-1.28789	-8.9600	~~1~~ 1.2721	-0.6426	~~1~~ 0.0636	-0.0321
3	-4.12840	.78400	-4.9000	-2.21282	-7.8400	~~1~~ 0.7271	-0.9493	~~1~~ 0.0364	-0.0475
4	-5.36878	.64799	-3.6001	-2.8777	-6.4799	~~1~~ 0.0021	-0.9657	~~1~~ 0.0001	-0.0483
5	-5.8429	.50000	-2.5001	-3.1318	-5.0000	-0.6319	-0.6319	-0.0316	-0.0316
6	-5.3838	.35200	-1.6000	-2.8857	-3.5200	-0.9657	-0.0021	-0.0483	~~1~~ 0.0001
7	-4.1216	.21600	-0.9001	-2.2092	-2.1600	-0.9493	~~1~~ 0.7271	-0.0475	~~1~~ 0.0364
8	-2.3929	.10400	-0.4000	-1.2826	-1.0400	-0.6426	~~1~~ 1.2721	-0.0321	~~1~~ 0.0636
9	-0.7580	.02799	- .1000	- .40629	- .2799	- .2264	~~1~~ 1.2069	-0.0113	~~1~~ 0.0603

$$M_L = M_o + \bar{y} H_o - V_o \bar{y}$$

APPENDIX E

Sample Calculations for Determining:

- (1) Absolute Stiffness
- (2) Carry-over Factor
- (3) Horizontal Thrust Due to Rotation of the Joint
- (4) Effect of Spread on End Reactions

Sample calculations for determining absolute stiffness, carry-over factor, horizontal thrust due to rotation of the joint, and effect of spread on end reactions are noted in the following text.

Data: 20' Span, 6' Rise, Rise/span - .30

Elastic Areas as previously computed.

Neutral Point - 4.0211' above springing line.

See Figure 4.

For our span:

$$E I_c \delta_{vm} = E I_c \alpha_{mv} = \frac{S}{2} \Sigma A = 10(20.0274) = 200.274$$

$$E I_c \delta_{hm} = E I_c \alpha_{mh} = \bar{y} \Sigma A = 4.0211(20.0274) = 80.5320$$

$$E I_c \alpha_{mm} = \Sigma A = 20.0274$$

$$E I_c \delta_{vh} = E I_c \delta_{hv} = \frac{S}{2} \bar{y} \Sigma A = 10(4.0211)(20.0274) = 805.320$$

$$E I_c \delta_{vv} = \Sigma A x^2 / \Sigma I_y \text{ c.g.} = 2663.2762 / 6.6756 = 2669.9518$$

$$E I_c \delta_{hh} = \Sigma A y^2 / \Sigma I_x \text{ c.g.} = 384.6722 / 3.1720 = 387.8442$$

It is essential to add the moments of inertia of the individual segments to the Ax^2 and Ay^2 values. An error of 4% arises by not using this method. If the conjugate structure is used the δ_{vv} and δ_{hh} as found contains these values and no computations are necessary for the moments of inertia of the arc segments. This is one of the advantages of the conjugate structure method.

Computations for Ax^2 , Ay^2 , and Ax

<u>Arc</u>	<u>A</u>	<u>x</u>	<u>x²</u>	<u>Ax²</u>	<u>y</u>	<u>y²</u>	<u>Ay²</u>	<u>Ay</u>
0-1	2.0010	1	1	2.0010	1.14	1.2996	2.6005	2.2811
1-2	2.0017	3	9	18.0153	3.06	9.3636	18.7431	6.1252
2-3	2.0026	5	25	50.0650	4.50	20.2500	40.5265	9.0117
3-4	2.0038	7	49	98.1862	5.46	29.8116	59.7365	10.9407
4-5	2.0046	9	81	162.3726	5.94	35.2836	70.7295	11.9073
5-6	2.0046	11	121	242.5566	5.94	35.2836	70.7295	11.9073
6-7	2.0038	13	169	338.6422	5.46	29.8116	59.7365	10.9407
7-8	2.0026	15	225	450.5850	4.50	20.2500	40.5265	9.0117
8-9	2.0017	17	289	578.4913	3.06	9.3636	18.7431	6.1252
9-10	2.0010	19	361	722.3610	1.14	1.2996	2.6005	2.2811
0-10	20.0274			2663.2762			384.6722	80.532

20' span, Rise/span = 0.30

Moment of Inertia of Arc Segment About Its Own Center of Gravity

<u>Sect.</u>	<u>A</u>	<u>x</u>	<u>y=ax</u>	<u>b</u> <u>va-yb</u>	<u>b²</u>	<u>a²</u>	<u>Iy c.g.</u> <u>Ab²/12</u>	<u>Ix c.g.</u> <u>Aa²/12</u>
0-1	2.0010	8	3.94	2.16	4.6656	4	.7780	.6670
1-2	2.0017	6	2.16	1.68	2.8224	4	.4716	.6672
2-3	2.0026	4	0.96	1.20	1.4400	4	.2403	.6675
3-4	2.0038	2	0.24	0.72	0.5184	4	.0865	.6679
4-5	2.0046	0	0.00	0.24	0.0576	4	.0096	.6682
							<u>1.5860</u>	<u>3.3378</u>

$$Iy \text{ c.g.} = 2 \times 1.586 = 3.172$$

$$Ix \text{ c.g.} = 2 \times 3.3378 = 6.6756$$

Substitute in equations(4), (5), and (6), found on page 10, the following:

$$EI\delta V = 200.2740 M_A + 2669.9518 V_A + 805.3218 H_A$$

$$EI\delta H = 80.5322 M_A + 805.3218 V_A + 387.8442 H_A$$

$$EI\theta = 20.0274 M_A + 200.274 V_A + 80.532178 H_A$$

But, since δV and δH are equal to zero, solving equation (4) and (5) simultaneously results in $V_A = -0.033130 M_A$ and $H_A = -0.13892 M_A$, in which H_A is the horizontal thrust due to rotation of the joint and is equal to $-0.83552 M_A/h$.

Using the free body diagram in the discussion and solving for M_B :

$$M_B = M_A + V_A S$$

$$M_B = M_A - 0.6626 M_A = 0.3374 M_A$$

$$M_B = 0.3374 M_A$$

By definition the carry-over factor is equal to the moment induced at the fixed end of a member when a unit moment is applied at the hinged end. Hence, the carry-over factor = -0.3374 .

The carry-over factor being negative since the induced moment at B causes tension on the same side of the member at the fixed end as the applied moment causes on the hinged end.

Substituting the values of M_A and H_A just obtained in equation (6), the following rotation at joint A occurs:

$$EI\theta = 2.1948 M_A$$

And when θ is equal to one radian,

$$M_A = 0.4556 EI \quad (\text{Absolute Stiffness})$$

$$M_A = 9.112 \frac{EI}{S} \quad (\text{Absolute Stiffness})$$

Effect of Spread

When spread occurs with no rotation or vertical deflection at the ends of the arch, equations (4) and (6) become equal to zero. Solving them simultaneously,

$$V_A = 0$$

$$H_A = -.248688 M_A$$

$$M_A = -4.021099 H_A$$

Substituting in equation (5) and solving for H_A and M_A in terms of $EI\delta$ DELTA H.

$$M_A = -0.06273 EI\delta \text{ DELTA H}$$

$$H_A = \nearrow .015621 EI\delta \text{ DELTA H}$$

Solving several span lengths, with the same rise/span ratio and with varied rise/span ratios, it was determined that M_A varied inversely as the product of the span and rise and that the thrust varied inversely as the product of span and the square of the rise. Therefore:

$$M_A = -7.5279 \frac{EI\delta \text{ DELTA H}}{Sh}$$

$$H_A = \nearrow 11.2471 \frac{EI\delta \text{ DELTA H}}{Sh^2}$$

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